Simulation of stationary process of electrochemical axisymmetric shaping

N.M. Sherykhalina
Department of Computer Science and Robotics
Ufa State Aviation Technical University
Ufa, Russia
e-mail: n_sher@mail.ru

N.I. Zhitnikova
Department of Computer Science and Robotics
Ufa State Aviation Technical University
Ufa, Russia
e-mail: zhitnikova_ni@mail.ru

L.I. Vasilyeva
Department of Computer Science and Robotics
Ufa State Aviation Technical University
Ufa, Russia
e-mail: lidav@mail.ru

A.A. Sokolova
Department of Computer Science and Robotics
Ufa State Aviation Technical University
Ufa, Russia
e-mail: alexandrakrasich@gmail.com**)**

Abstract[1]

The problem of stationary formation and the boundary conditions are formulated according the Faraday low. The problem is reduced to solving of a boundary-value problem for determining of the analytic function of a complex variable. In contrast to the plane problem the integral transformations of the analytic function are applied to determine the tension intensity. A spline functions approximation is fulfill, the algorithms for general solution of stationary axisymmetric problems are described. The results of the numerical solution with great accuracy are presented.

1. Introduction

Electrochemical dimensional machining (ECM) is one of the most promising ways of obtaining complex shape parts for hard work materials. In addition there is practically no force or heat effect on the workpiece with the ECM, electrode- tool (ET) practically does not wear out. However, since machining is carried out in a non-contact mode and, unlike electro-erosion machining, the shape of the part is not equidistant to the surface of the ET, the formation calculation of the machined surface is a complex problem [1].

The known methods for such problems solving, unfortunately, do not have sufficient accuracy and stability to the accumulation of round off errors in the calculation.

In this connection the goal of this paper is development of a numerical-analytical method and investigation of the characteristics of stationary processes.

2. The problem statement

2.1. Mathematical model of the process

We consider the problem of the Laplace equation solving for a potential Φ inside a certain axisymmetric domain. The condition of constancy of Φ fulfills at the domain boundaries, and the form of the free boundary in the zone of high current densities must satisfy the stationarity condition, i.e. the equality of the projection of the speed of motion of the ET on the external normal to the machined surface and rate of electrochemical dissolution (Fig.1). The conditions on other parts of the machined surface are discussed below.

Let us consider the stationary problem of machining with a point tool electrode. The meridional cross section of the interelectrode space (IES) is shown in Fig. 1. Here *ADB* is the boundary of the dissolved material, the point *C* is the point ET moving with the velocity V_{et} to the machined surface.

The potential Φ and the stream function Ψ of an axisymmetric field are expressed in terms of function of a complex variable. It is analytical (satisfying the Cauchy-Riemann conditions) in the domain $Z = X + iY$, whose shape coincides with the shape of the boundaries of the interelectrode space in the meridional section of the axisymmetric field with help of the formulas (the Polozhy integral transformations [2]):

$$\Phi(x,y) = \frac{1}{\pi}\int_{-\infty}^{z} \frac{f(Z)\,dZ}{\sqrt{(Z-z)(Z-\bar{z})}}, \quad (1)$$

[1]**Proceedings of the 6th All-Russian Scientific Conference "Information Technologies for Intelligent Decision Making Support", May 28-31, Ufa - Stavropol, Russia, 2018**

$$\psi(X_0,Y_0)=\frac{1}{\pi}\,\mathrm{Im}\int_{-\infty}^{Z_0} f(Z)\frac{(Z-X_1)}{\sqrt{(Z-Z_0)(Z-\bar{Z}_0)}}dZ, \quad (2)$$

where $Z_0=X_0+iY_0$, $\bar{Z}_0=X_0-iY_0$.

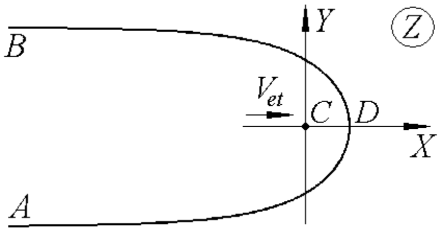

Fig 1. The diagram of IES for the stationary process

Thus, the solution of the axisymmetric problem is reduced to solving a certain plane problem for the determination of an analytic function $W(Z)$ that represents the complex potential of some auxiliary plane field. The potential and the stream function of the axisymmetric field are obtained by integral transformations (1), (2) applied to the function *f*(*Z*)=*dW*/*dZ* [1].

The boundary conditions of the auxiliary plane problem are written in the form of integral equations which are obtained by equating of the right-hand sides (1) for equipotential boundaries to a constant or (2) for impermeable ones. The equality to zero of the imaginary or real part of *f(Z)* in the general case does not lead to equality to zero or a constant of the corresponding integrals.

The values of the tangential and radial component of the tension are determined from (1) and (2)

$$E_x=\frac{\partial\Phi}{\partial X}=\frac{1}{\pi}\,\mathrm{Im}\int_{X_1+i0}^{X_0+iY_0}\frac{f\,dZ}{\sqrt{(Z-Z_0)(Z-\bar{Z}_0)}}, \quad (3)$$

$$E_y=\frac{\partial\Phi}{\partial Y}=\frac{1}{\pi Y}\,\mathrm{Im}\int_{X_1+i0}^{X_0+iY_0}\frac{f(Z)(Z-X_1)\,dZ}{\sqrt{(Z-Z_0)(Z-\bar{Z}_0)}}, \quad (4)$$

where X_1+i0 is certain point on symmetry axis *x*.

We conformally map the domain corresponding to IES on the plane *Z* on the band $\chi=\sigma+i\upsilon$ (fig. 2,a). In this case the problem of determining of the function *W (Z)* analytical in the IES domain can be solved in a parametric form. So, we find *W*(χ) and *Z*(χ).

2.2. The boundary conditions

The boundary condition for determining of the function *W*(χ) is the condition for the anode equipotentiality. The image of the IES domain is a curvilinear half-band on the *W* plane (Fig. 2, b).

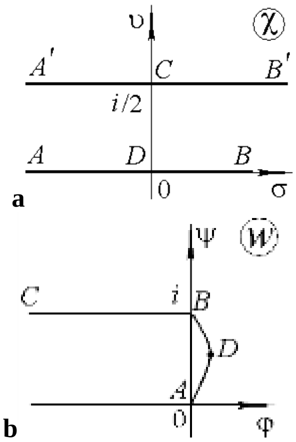

Fig 2. IES images on the planes: a – parametric variable χ; b – complex potential

The boundary condition for determining of the function *Z*(χ) is the stationarity condition. The Faraday law in the form of the Polubarinova-Galin equation [1, 3, 4] is used to obtain this condition in the axisymmetric case

$$\mathrm{Im}\left(\frac{\partial\bar{Z}}{\partial t}\frac{\partial Z}{\partial\sigma}\right)=k\eta\frac{1}{Y}\frac{\partial\Psi}{\partial\sigma}, \quad (5)$$

where *k* is electrochemical constant; η is a current efficiency which is to be constant for a stationary process.

Let us consider the coordinate system associated with the ET. Then the surface of the anode moves along with the tool electrode with the velocity V_{et}. We move to the mobile system by replacing the variable $Z(\chi,t)=Z_1(\chi)+V_{et}t$ in the formula (5)

$$\mathrm{Im}\left[V_{et}\frac{\partial\bar{Z}_1}{\partial\sigma}\right]=k\eta\frac{1}{Y}\frac{\partial\Psi}{\partial\sigma}. \quad (6)$$

If $V_{et}=|V_{et}|$ then (6) takes the form [5]

$$-V_{et}\frac{\partial Y}{\partial\sigma}=k\eta\frac{1}{Y}\frac{\partial\Psi}{\partial\sigma}. \quad (7)$$

Integrating (7) we have

$$V_{et}\frac{Y^2(\sigma)}{2}=k\eta\left[\Psi(\sigma)-\Psi(-\infty)\right]. \quad (8)$$

It is more convenient to use dimensionless magnitudes in calculations

$$z=\frac{Z}{l},\ x=\frac{X}{l},\ y=\frac{Y}{l},\ w=\frac{W}{Ul},$$

where *l* is character size; $U=I/(\kappa l)$, *I* is current; κ is electrolyte conductivity.

According to the Faraday low the quantity of dissolved metal is equal to

$$\frac{k\eta}{\kappa}IQ=\frac{k\eta}{\kappa}\Psi=V_{et}S,$$

where Q is a charge flowing in a circuit during Δt, S is a cross-sectional square of the groove formed with deepening of the ET into the body of the workpiece $S = \pi R^2$; R is a radius of the groove. Then we obtain the formula

$$\pi R^2 = \frac{k\eta I}{\kappa V_{et}}, \quad l = 2R = \frac{2}{\sqrt{\pi}}\sqrt{\frac{k\eta I}{\kappa V_{et}}}. \qquad (9)$$

And also the dimensionless velocity of the ET $v_{et} = 1$. In dimensionless variables the stationarity condition takes the form

$$\frac{2}{\pi} y^2(\sigma) = \psi(\sigma) - \psi(0). \qquad (10)$$

3. The method of problem solving

The method of solving of axisymmetric problems for determining of forms that are not time dependent includes two main steps: the finding of a conformal mapping of a parametric variable area onto a physical plane and the determining of the potential and stream function using integral transforms of an analytic function.

3.1. The conformal mapping

The conformal mapping problem is solved as follows. It is convenient to choose a band of width 1/2 with the correspondence of the points indicated in fig. 2,a as the area of variation of the parametric variable χ=σ+iυ.

The function mapping the plane χ on the physical one is found as the sum

$$z(\chi) = z_0(\chi) + z_\Delta(\chi). \qquad (11)$$

The quantity $\mathrm{Re}\, z_\Delta(\chi) \to 0$ for $\chi \to \infty$. The function

$$z_0 = \frac{2}{\pi}\ln \operatorname{ch}\left[\pi\left(\chi - \frac{i}{2}\right)\right] =$$

$$= -\frac{1}{\pi}\ln\frac{\operatorname{ch}\pi\sigma}{2} + \frac{i}{\pi}\operatorname{arsh}(\ldots) \qquad (12)$$

is the solution of the plane problem [1]. Then the boundary χ=σ+*i*0 is mapped onto the surface *ADB*, the boundary χ=σ+*i*/2 is mapped onto the cut *A′C′B*. The location of the point source is $z_0(i/2) = 0$. The derivatives

$$\frac{dz_0}{d\chi} = \operatorname{th}\left[\frac{\pi}{2}\left(\chi - \frac{i}{2}\right)\right] - \frac{i}{2}\,\frac{\operatorname{ch}\pi\chi}{2\operatorname{ch}^2\left[\frac{\pi}{2}\left(\chi - \frac{i}{2}\right)\right]},$$

$$\frac{dz_0}{d\sigma} = -\left(\operatorname{th}\pi\sigma - \frac{i}{\operatorname{ch}\pi\sigma}\right). \qquad (13)$$

The function $z_\Delta(\chi)$ is determined as follows. We find the solution on the boundary χ=σ at the node points σ_m(m=0,…,n). The required values are $\mathrm{Im}\, z_\Delta(\sigma_m) = y_m$.

We put $\mathrm{Im}\, z_\Delta(\sigma_n) = 0$ for $\sigma = \sigma_n$ as $\mathrm{Im}\, z_\Delta(\sigma)$ quickly (as exponent) decreases for σ→∞. The values $\mathrm{Im}\, z_\Delta(\sigma)$ at intermediate between the node points we find similarly to [6,7], using the cubic spline *P*, which has two continuous derivatives.

To restore the function $z_\Delta(\chi)$ we use the Schwarz formula [8] taking into account that $z_\Delta(\chi)$ is analytic function that has like $z_0(\chi)$ the only real values on the straight line Imχ=1/2. Analytically continuing the function $z_\Delta(\chi)$ to a band of unit width we get

$$z_\Delta(\chi) = \int_0^\infty P(\sigma)\frac{\operatorname{sh}2\pi\chi\, d\sigma}{\operatorname{ch}^2\pi\chi - \operatorname{ch}^2\pi\sigma}. \qquad (14)$$

The derivative $\frac{dz_\Delta}{d\chi}(\chi)$ is defined by differentiation (14)

$$\frac{dz_\Delta}{d\chi}(\chi) = \operatorname{sh}2\pi\chi\int_0^\infty \frac{dP}{d\sigma}(\sigma)\frac{d\sigma}{\operatorname{ch}^2\pi\chi - \operatorname{ch}^2\pi\sigma}. \qquad (15)$$

Note, according to (12), (14) $z_0(i/2) = 0$, and

$$z_\Delta(i/2) = 2\int_0^\infty P(\sigma)\operatorname{th}\pi\sigma\, d\sigma. \qquad (16)$$

3.2. Potential and stream function determination

The axisymmetric problem is solved by reducing it to an auxiliary plane problem. So, the area corresponding to the IES on the plane of the complex potential (Fig. 2, b) is conformally mapped onto the plane of the parametric variable χ (Fig. 2, a).

The solution consists of representing of the complex potential $w(\chi)$ of the auxiliary plane problem and its derivatives in the form of sums

$$w(\chi) = w_0(\chi) + w_1(\chi),$$

$$\frac{\partial w}{\partial \chi}(\chi) = f_0(\chi) + f_1(\chi).$$

where $f_0(\chi) = \frac{i}{\operatorname{ch}\pi\chi}$ is determined from the solution of the plane problem.

We find the solution in the form of the function

$$f_1(\chi) = \frac{\partial w_1}{\partial \chi}(\chi).$$

This function must have the certain properties: its real part have to be an odd function of σ for χ=σ+*i*0, $f_1(\sigma + i/2)$ have to be real for χ=σ+*i*/2. Then it can be analytically continued to a band of unit width. Then $\mathrm{Re} f_1(\sigma+i) = \mathrm{Re} f_1(\sigma+i0)$.

The required parameters are the values of the real part of the function $\operatorname{Re} f_1(\sigma_m) = f_m$ at node points σ_m, $(m=1,\dots,n)$. As far as f_1 is the odd function of σ, $\operatorname{Re} f_1(\sigma_0)=0$ for $\sigma=\sigma_0=0$. Let $\operatorname{Re} f_1(\sigma_n)=0$ for $\sigma=\sigma_n$ because $f_1(\sigma)$ rapidly (as exponent) decreases for $\sigma\to\infty$. The values $\operatorname{Re} f_1(\sigma)$ at intermediate between the node points is found using a cubic spline $S(\sigma)$.

The Schwartz formula [8] is applied to restore the function $f_1(\chi)$. As the function $\operatorname{Re} f_1(\sigma)$ is odd with respect to σ then

$$f_1(\chi) = -i\frac{2\chi}{\pi}\int_0^\infty \frac{S(\sigma)\,d\sigma}{\sigma^2-\chi^2}.$$

It is necessary to integrate from infinity at applying of the Polozhy transformations in connection with the presence of feature of the function $f(\sigma)$ at point C. The expressions (1), (2) take the form

$$\Theta(\sigma_m) = \frac{1}{\pi}\int_{\sigma_m}^{\sigma_n} \ldots \frac{d\sigma}{\sqrt{(z-z(\sigma_1))(z-z(\sigma_n))}}, \quad (17)$$

$$\Theta(\sigma_m) = \frac{1}{2\pi}\int_{\sigma_m}^{\sigma_n} \ldots \frac{(z-z(\sigma))\,d\sigma}{\sqrt{(z-z(\sigma_1))(z-z(\sigma_n))}}. \quad (18)$$

The condition of equipotentiality of the machined surface for $\chi=\sigma_0$ in the collocation method leads to the system of equations

$$F_m = \Theta(\sigma_m) - \Theta(\sigma_n) = 0, \quad m=1,\dots,n-1. \quad (19)$$

Substituting in (17) the expression $f_1(\sigma)$ by the spline and the Schwarz formula, substituting the obtained expressions in (19), and using the Sokhotsky formula [8], we obtain the system of linear (with respect to the variables f_m) equations

$$F_m = \frac{1}{\pi}\int_{\sigma_m}^{\sigma_n}\left\{ \ldots \int_0^\infty \frac{S(\tau)\,d\tau}{\tau^2-\sigma^2} \right\} \times \frac{d\sigma}{\sqrt{(z-z(\sigma_m))(z-z(\sigma_n))}} = 0.$$

It is necessary to find partial derivatives $\frac{\partial F_m}{\partial f_j}$ to form this system of linear equations and to differentiate the spline $S(\sigma)$ under the integral sign. We obtain the unit spline $E_j(\sigma)$ differentiating the spline $S(\sigma)$ with respect to f_j. Then

$$\frac{\partial F_m}{\partial f_j} = \frac{1}{\pi}\int_{\sigma_m}^{\sigma_n}\left\{ \ldots \int_0^\infty \frac{E_j(\tau)\,d\tau}{\tau^2-\sigma^2} \right\} \times \frac{d\sigma}{\sqrt{(z-z(\sigma_m))(z-z(\sigma_n))}}.$$

Thus, the system of equations (19) takes the form

$$\sum_{j=1}^{N}\frac{\partial F_m}{\partial f_j} f_j = B_m, \quad m=0,\dots,N, \quad (20)$$

$$B_m = \frac{1}{\pi}\int_{\sigma_m}^{\sigma_n} \ldots \frac{d\sigma}{\sqrt{(z-z(\sigma_m))(z-z(\sigma_n))}}.$$

The problem is solved by the collocation method. The required values are $\operatorname{Im} z_\Delta(\sigma_m) = y_m$, $\operatorname{Re} f_1(\sigma_m) = f_m$ at node points σ_m ($m=1,\dots, n-1$). (As it is shown above $y_0=y_n=0$, $f_0=f_n=0$). The system of nonlinear equations is constructed for determination of these parameters. It is required the fulfilling of equations (10), (20) for $\sigma=\sigma_m$ ($m=1,\dots,n-1$). The maximum value of σ_n is equal to 10.

Thus, we have the nonlinear equations system that is solved by the Newton method with step regulation.

4. Numerical results

The shapes of machined surface are shown in Fig. 3-5 as well as the dependence of the curvature and strength on the ordinate of the machined surface in comparison with the plane case, the solution of which is obtained by the formula [1]

$$x = \frac{1}{\pi}\ln 2\cos \pi y.$$

The asymptotic dependence for the plane problem is $\frac{1}{2} - y \sim e^{\pi x}$, for the axisymmetric problem is $\frac{1}{2} - y \sim e^{1.53\pi x}$ (according to the computed data).

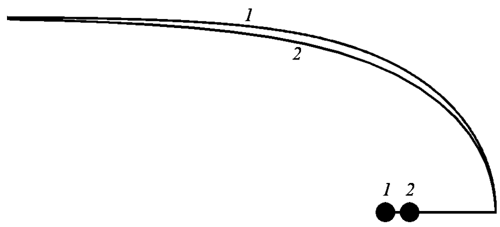

Fig 3. The stationary shapes for the axisymmetric (the curve 1) and the plane (curve 2) problems

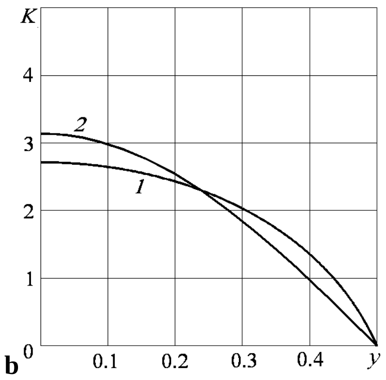

Fig 4. The dependence of curvature on ordinate for the axisymmetric (the curve 1) and the plane (curve 2) problems

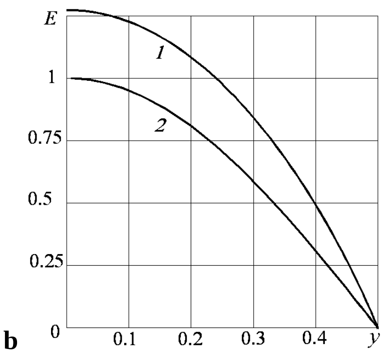

Fig 5. The dependence of dimensionless strength on ordinate for the axisymmetric (the curve 1) and the plane (curve 2) problems

The values of solution parameters with error estimate and relative diffusion are in the table 1. Here x_D, K_D, E_D are abscissa, dimensionless strength and curvature at the point *D*. The corresponding illustrations of estimates and numerical filtration are presented in Fig.6.

Table 1. Results of the stationary axisymmetric problem

Parameter	Value	Error estimate	Relative diffusion
x_D	0.282186858398	$\pm 1\cdot 10^{-12}$	0.1
E_D	4/π	$\pm 1\cdot 10^{-9}$	0.1
K_D	2.716660	$\pm 1\cdot 10^{-8}$	0.1

The error estimate of the numerical solution is carried out by filtration of the calculations results [9-12]. The results of filtration are presented on a logarithmic scale in fig. 6. The ordinate axis shows the decimal logarithms of the absolute values of the obtained error estimates $-\lg\Delta$, (the accuracy of the obtained data). The decimal logarithms of the segments number of *n* (which varies from 64 to 1339) are plotted on the abscissa axis. The values of anode strength E_D and the surface curvature K_D at the central point D are considered as the estimated parameters in Fig. 6. Digit 0 marks the estimated accuracy of the calculated data, digits 1, 2, ... marks the results of the first and second, etc., filtration. The ordinate difference between the two curves represents the logarithm of the ratio of the estimates for different filtration. This ratio is called relative diffusion. At the level of the 10th digit of the value E_D, there is a violation of regularity caused by the round off error. In the curvature dependence the violation of regularity is observed at the level of the 9th digit, since the second derivative of the required function is used to determine it.

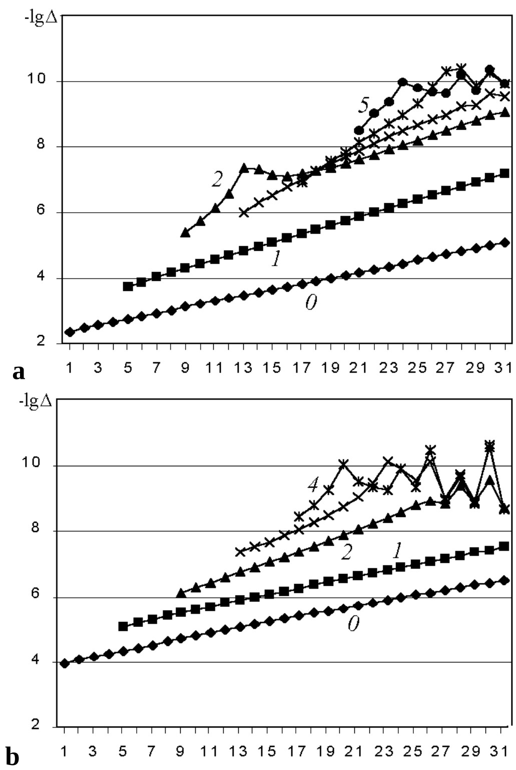

Fig 6. The relative error estimate of axisymmetric problem solution: a – of the strength E_D at the point *D*; б – of the curvature K_D at the point *D*

4. Conclusion

Thus, the numerical solution method of the problem of the stationary electrochemical machining by a point tool electrode in an axisymmetric formulation is proposed in this paper. The method is based on integral transformations of the analytical function. Numerical solution confirmed the high efficiency of the proposed method. Numerical values (with an error estimate) of geometrical and physical parameters are obtained. For example, the curvature of the boundary is calculated with an accuracy of 8 significant digits..:

Acknowledgments

The work is financially supported by the RFBR grant (project 17-07-00356).

References

1. Zhitnikov V.P., Zaytsev A.N. “Pulse electrochemical dimensional machining”. Mashinistroenie, Moscow, 2008.

2. Pology G.N. “Generalization of the theory of analytic functions of a complex variable”. Kiev Univ:, Kiev, 1965.
3. Polubarinova-Kochina P.Y. “Nonstationary moving at filtration theory”. *Applied Mathematics and mechanics*,. 1945, 9: 79-90.
4. Galin L.A. “Nonstationary filtration with free boundaries”. *DAN USSR*, 1945, 47: 246-249.
5. Sherykhalina N.M., Zinnatullina O.R., Sokolova A.A. “Simulation of stationary process of electrochemical axisymmetric shaping by the point electrode-tool”. *Uspehi sovremennoy nauki*, Belgorod, 2017, 2 (8): 137 – 144.
6. Zhitnikov V.P., Fedorova G.I., Sherykhalina N.M., Urakov A.R. “Numerical investigation of non-stationary electrochemical shaping based on an analytical solution of the Hele-Shaw problem” *Journ. Eng. Math*., 2006, 55. (1-4):. 255-276.
7. Zhitnikov V.P., Zinnatullina O.R., Porechny S.S., Sherykhalina N.M. “Determining the limiting solutions of nonstationary axisymmetric Hele-Shaw problems”. *Journal of Applied Mechanics and Technical Physics*, 2009,. 50 (4): 617-627.
8. Lavrent’ev M.A., Shabat B.V. “Methods of the theory of functions of complex variable”. Nauka, Moscow,. 1973.
9. Zhitnikov V.P., Sherykhalina N.M. “Methods of verification of mathematical models in conditions of uncertainty”. *Vestnik UGATU*, 2000, 2: 53-60.
10. Zhitnikov V.P., Sherykhalina N.M. Justification of methods of filtration of numerical experiment results”. *Vestnik UGATU*, 2007, 9 (3): 71-79.
11. Zhitnikov V.P., Sherykhalina N.M., Porechny S.S. “About one approach to practical error estimation of numerical results”. *Scientific-technical sheets of SPbGTU*, Saint-Petersburg, 2009, 3 (80): 105-110.
12. Zhitnikov V. P., Sherykhalina N.M., Sokolova A.A. “Problem of Reliability Justification of Computation Error Estimates”. *Mediterranean Journal of Social Sciences*, 2015, 6, (2): 65 - 78.